KASR TARTIBLI DIFFUZIYA TENGLAMASI UCHUN KOSHI MASALASINI YECHISH USULLARI

Badalov Tolib Tohir o‘gli
Osiyo Xalqaro Universiteti

Annotatsiya. *Ushbu maqolada vaqt bo‘yicha Caputo ma’nosidagi kasr tartibli hosilaga ega bo‘lgan diffuziya tenglamasi uchun Koshi masalasini yechish usullari tadqiq etiladi. Masalaning analitik yechimini topishda Furye va Laplas integral almashtirishlari usuli qo‘llanilgan hamda yechim Wright va Foxning H - funksiyalari orqali integral ko‘rinishida ifodalangan. Maqolada asosiy operatorlarning xossalari, teoremalarning qat’iy matematik isbotlari va olingan natijalarning amaliy tatbiqi sifatida aniq model masalalar yechib ko‘rsatilgan. Natijalar anomal diffuziya va relyaksatsiya jarayonlarini modellashtirishda muhim ahamiyatga ega.*

Kalit so‘zlar: *kasr tartibli hosila, Caputo hosilasi, Koshi masalasi, Laplas almashtirishi, Furye almashtirishi, Wright funksiyasi, anomal diffuziya, fundamental yechim.*

Аннотация. *В данной статье исследуются методы решения задачи Коши для уравнения диффузии с дробной производной по времени в смысле Капуто. Для нахождения аналитического решения задачи применен метод интегральных преобразований Фурье и Лапласа, а решение выражено в интегральной форме через функции Райта и H-функции Фокса. В работе подробно изложены свойства основных операторов, строгие математические доказательства теорем, а также в качестве практического применения полученных результатов решены конкретные модельные задачи. Результаты имеют важное значение для моделирования процессов аномальной диффузии и релаксации.*

Ключевые слова: *дробная производная, производная Капуто, задача Коши, преобразование Лапласа, преобразование Фурье, функция Райта, аномальная диффузия, фундаментальное решение.*

Abstract *This paper investigates the methods for solving the Cauchy problem for the diffusion equation with a fractional time derivative in the Caputo sense. To find the analytical solution of the problem, the Fourier and Laplace integral transform methods are applied, and the solution is expressed in integral form through Wright functions and Fox's H-functions. The paper presents in detail the properties of the main operators, rigorous mathematical proofs of theorems, and specific model problems are solved as a practical application of the obtained results. The results are of great importance for modeling anomalous diffusion and relaxation processes.*

Keywords: *fractional derivative, Caputo derivative, Cauchy problem, Laplace transform, Fourier transform, Wright function, anomalous diffusion, fundamental solution.*

So‘nggi o‘n yilliklarda kasr tartibli differensial tenglamalar nazariyasi matematik fizika, mexanika, gidrodinamika va fraktal muhitlar fizikasining eng tez rivojlanayotgan sohalaridan biriga aylandi. Klassik butun tartibli diffuziya tenglamasi zarrachalarning Markov jarayonlari va Broun harakatini tavsiflashda cheklangan imkoniyatlarga ega bo‘lib, g‘ovakli va tartibsiz muhitlarda, polimer tizimlarida hamda plazma fizikasida kuzatiladigan anomal diffuziya (subdiffuziya va superdiffuziya) hodisalarini adekvat modellashtira olmaydi. Ushbu fizik jarayonlarning tabiati “xotira effekti” (memory effects) va uzoq masofali korrelyatsiyalar bilan xarakterlanadi, bularni matematik nuqtai nazardan kasr tartibli integro-differensial operatorlar yordamida bayon etish o‘rinlidir. Vaqt bo‘yicha kasr tartibli hosilaga ega bo‘lgan subdiffuziya tenglamasi uchun Koshi masalasini o‘rganish bu boradagi fundamental muammolardan biri hisoblanadi.

Ushbu tadqiqotda biz vaqt bo‘yicha $0 < \alpha \leq 1$ tartibli Caputo kasr hosilasiga ega bo‘lgan quyidagi anomal diffuziya tenglamasini ko‘rib chiqamiz:

$$ {}^{C}\partial_{0t}^{\alpha} u(x,t) = \lambda \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0, $$

bu yerda $\lambda > 0$ — diffuziya koeffitsiyenti, ${}^{C}\partial_{0t}^{\alpha}$ esa quyidagi ko‘rinishga ega bo‘lgan Caputo ma’nosidagi kasr tartibli hosiladir:

$$ {}^{C}\partial_{0t}^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u(x,\tau)}{\partial \tau} d\tau. $$

Berilgan tenglama uchun boshlang‘ich shart (Koshi sharti) quyidagicha qo‘yiladi:

$$ u(x,0) = \varphi(x), \quad x \in \mathbb{R}, $$

bunda $\varphi(x)$ — berilgan yetarlicha silliq va abssolyut integrallanuvchi funksiyadir. Masalaning cheksiz doirada qaralayotganini hisobga olib, cheksizlikda yechimning so‘nuvchanlik shartini talab qilamiz: $\lim_{|x|\to\infty} u(x,t) = 0$.

Ushbu Koshi masalasini analitik yechish uchun integral almashtirishlar usulidan, xususan, fazoviy o‘zgaruvchi x bo‘yicha Furye almashtirishi va vaqt o‘zgaruvchisi t bo‘yicha Laplas almashtirishining kombinatsiyasidan foydalanamiz. Berilgan funksiya $f(x)$ ning Furye almashtirishi va uning teskarisi mos ravishda quyidagicha aniqlanadi:

$$ \hat{f}(\xi) = \mathcal{F}\{f(x)\}(\xi) = \int_{-\mathbb{R}} f(x) e^{-i\xi x} dx, \quad f(x) = \mathcal{F}^{-1}\{\hat{f}(\xi)\}(x) = \frac{1}{2\pi} \int_{-\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi. $$

Shuningdek, $g(t)$ funksiyaning Laplas almashtirishi quyidagi integral orqali ifodalanadi:

$$ \tilde{g}(s) = \mathcal{L}\{g(t)\}(s) = \int_0^\infty g(t) e^{-st} dt. $$

Caputo kasr tartibli hosilasining Laplas almashtirishi xossasiga ko‘ra,

$$ \mathcal{L}\{ {}^{C}\partial_{0t}^{\alpha} u(x,t)\}(s) = s^{\alpha} \tilde{u}(x,s) - s^{\alpha-1} u(x,0) $$

munosabat o‘rinlidir. Dastlab, Koshi masalasining har ikkala tomoniga x o‘zgaruvchisi bo‘yicha Furye almashtirishini qo‘llaymiz. Furye almashtirishining hosila olish qoidasiga ko‘ra, $\mathcal{F}\{\frac{\partial^2 u}{\partial x^2}\} = -\xi^2 \hat{u}(\xi,t)$ bo‘ladi. Natijada, kasr tartibli differensial tenglama quyidagi ko‘rinishga keladi:

$$^{C}\partial_{0t}^{\alpha}\hat{u}(\xi,t) = -\lambda\xi^2\hat{u}(\xi,t), \quad \hat{u}(\xi,0) = \hat{\varphi}(\xi).$$

Hosil bo'lgan oddiy kasr tartibli differensial tenglamaga vaqt o'zgaruvchisi t bo'yicha Laplas almashtirishini qo'llaymiz. Bu esa bizga quyidagi algebraik munosabatni beradi: $s^{\alpha}\tilde{\hat{u}}(\xi,s) - s^{\alpha-1}\hat{\varphi}(\xi) = -\lambda\xi^2\tilde{\hat{u}}(\xi,s)$.

Ushbu tenglamadan noma'lum $\tilde{\hat{u}}(\xi,s)$ funksiyani ajratib olamiz:

$$\tilde{\hat{u}}(\xi,s) = \frac{s^{\alpha-1}}{s^{\alpha}+\lambda\xi^2}\hat{\varphi}(\xi).$$

Endi teskari Laplas almashtirishini bajarish talab etiladi. Buning uchun Mittag-Leffler tipidagi maxsus funksiyalar nazariyasiga murojaat qilamiz. Ma'lumki, ikki parametrli Mittag-Leffler funksiyasi quyidagi qator ko'rinishida aniqlanadi:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta > 0,\ z \in \mathbb{C}.$$

Agar $\beta = 1$ bo'lsa, u bir parametrli Mittag-Leffler funksiyasi $E_{\alpha}(z) = E_{\alpha,1}(z)$ ga aylanadi. Mittag-Leffler funksiyasining Laplas almashtirishi jadval formulasiga ko'ra:

$$\mathcal{L}\{E_{\alpha}(-\omega t^{\alpha})\}(s) = \frac{s^{\alpha-1}}{s^{\alpha}+\omega}.$$

Bizning holatda$\omega = \lambda\xi^2$ekanligini hisobga olsak, teskari Laplas almashtirishini qo'llash natijasida quyidagi ifoda hosil bo'ladi:

$$\hat{u}(\xi,t) = E_{\alpha}(-\lambda\xi^2 t^{\alpha})\hat{\varphi}(\xi).$$

Keyingi bosqichda teskari Furye almashtirishini qo'llash orqali masalaning yechimini original koordinatalarda topamiz. Konvolutsiya (o'rama) haqidagi teoremaga asosan, ikki funksiya ko'paytmasining teskari Furye almashtirishi ularning asl nusxalarining konvolutsiyasiga teng:

$$u(x,t) = \int_{-\infty}^{\infty} G\,(x-\xi,t)\varphi(\xi)d\xi,$$

bu yerda $G(x,t)$ funksiya Koshi masalasining fundamental yechimi (Grin funksiyasi) bo'lib, u quyidagicha aniqlanadi:

$$G(x,t) = \mathcal{F}^{-1}\{E_{\alpha}(-\lambda\xi^2 t^{\alpha})\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\,E_{\alpha}(-\lambda\xi^2 t^{\alpha})d\xi.$$

$E_{\alpha}(-\lambda\xi^2 t^{\alpha})$ funksiyaning ξ o'zgaruvchisiga nisbatan juftligidan foydalanib, integralni quyidagicha yozish mumkin:

$$G(x,t) = \frac{1}{\pi}\int_{0}^{\infty}\cos\quad(\xi x)E_{\alpha}(-\lambda\xi^2 t^{\alpha})d\xi.$$

Ushbu integralni hisoblash va uni tahlil qilish uchun quyidagi muhim teoremani keltiramiz va uni qat'iy isbotlaymiz.

Teorema 1. Agar $0 < \alpha \leq 1\,va\,\lambda > 0$bo'lsa, u holda kasr tartibli diffuziya tenglamasi uchun Koshi masalasining fundamental yechimi quyidagi Wright funksiyasi orqali ifodalanadi:

$$G(x,t) = \frac{1}{2\sqrt{\lambda}t^{\alpha/2}}M_{\alpha/2}\left(\frac{|x|}{\sqrt{\lambda}t^{\alpha/2}}\right),$$

bu yerda $M_\mu(z)$ (M-Wright funksiyasi) quyidagi qator ko‘rinishiga ega:

$$M_\mu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma(-\mu k + 1 - \mu)}.$$

Isbot. Mittag-Leffler funksiyasining qatorli yoyilmasini fundamental yechim formulasiga qo‘yamiz va hadma-had integrallash amalini bajaramiz (bunga qatorning tekis yaqinlashuvchanligi kafolat beradi):

$$G(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos\quad(\xi x)\sum_{k=0}^{\infty} \frac{(-\lambda\xi^2 t^\alpha)^k}{\Gamma(\alpha k + 1)}d\xi = \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(\alpha k + 1)}\int_0^{\infty} \xi^{2k}\cos(\xi x)d\xi.$$

Biroq, ushbu integral klassik ma’noda uzoqlashuvchi bo‘lganligi sababli, Wright funksiyasining integral ifodalaridan yoki Mellin almashtirishidan foydalanish maqsadga muvofiqdir. Muqobil yo‘l sifatida, $G(x,t)$ ning Laplas tasviri $\tilde{G}(x,s)$ ni ko‘rib chiqamiz:$\tilde{G}(x,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{s^{\alpha-1}}{s^\alpha + \lambda\xi^2} e^{i\xi x}d\xi$. Ushbu integralni qoldiqlar (reziduumlar) haqidagi teorema yordamida hisoblaymiz. Maxrajning ildizlari $\xi = \pm i\frac{s^{\alpha/2}}{\sqrt{\lambda}}$ nuqtalarda joylashgan. Yuqori yarim tekislikdagi qutb nuqtani olib, integralni hisoblasak:

$$\tilde{G}(x,s) = \frac{s^{\alpha-1}}{2\pi}\cdot 2\pi i \cdot Res\left(\frac{e^{i\xi x}}{(\xi - is^{\alpha/2}/\sqrt{\lambda})(\xi + is^{\alpha/2}/\sqrt{\lambda})}, i\frac{s^{\alpha/2}}{\sqrt{\lambda}}\right) = \frac{s^{\alpha/2-1}}{2\sqrt{\lambda}}e^{-\frac{|x|}{\sqrt{\lambda}}s^{\alpha/2}}$$

Endi ushbu ifodadan teskari Laplas almashtirishini topishimiz kerak. Buning uchun quyidagi ma’lum integral munosabatdan foydalanamiz:

$$\mathcal{L}\{t^{-1}W_{\rho,0}(-zt^{-\rho})\} = s^{\rho-1}e^{-zs^\rho}.$$

Bizning holatda $\rho = \alpha/2\$va\$z = \frac{|x|}{\sqrt{\lambda}}$. Teskari Laplas almashtirishini bajarib, quyidagiga ega bo‘lamiz:

$$G(x,t) = \frac{1}{2\sqrt{\lambda}}t^{-1}W_{\alpha/2,0}\left(-\frac{|x|}{\sqrt{\lambda}}t^{-\alpha/2}\right).$$

Wright funksiyasining xossasiga ko‘ra, $W_{\mu,0}(-z) = \mu z M_\mu(z)$ munosabati o‘rinli bo‘lib, u bizni bevosita Teorema tasdig‘iga olib keladi:

$$G(x,t) = \frac{1}{2\sqrt{\lambda}t^{\alpha/2}}M_{\alpha/2}\left(\frac{|x|}{\sqrt{\lambda}t^{\alpha/2}}\right).$$

Teorema to‘liq isbotlandi.

Ushbu fundamental yechim klassik diffuziya tenglamasining fundamental yechimi bilan solishtirilganda umumiyroq xarakterga ega. Agar $\alpha = 1$ bo‘lsa, $M_{1/2}(z) = \frac{1}{\sqrt{\pi}}e^{-z^2/4}$ ekanligidan, klassik Gauss taqsimoti (Grin funksiyasi) kelib chiqadi: $G(x,t)|_{\alpha=1} = \frac{1}{2\sqrt{\pi\lambda t}}e^{-\frac{x^2}{4\lambda t}}$.

Olingan nazariy natijalarni mustahkamlash uchun aniq bir amaliy misolni ko‘rib chiqamiz.

Misol 1. Quyidagi kasr tartibli diffuziya tenglamasi uchun Koshi masalasini qaraylik:

$${}^{C}\partial_{0t}^{1/2}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2},\quad x \in \mathbb{R},\ t > 0,$$

boshlang‘ich shart esa Delta funksiya ko‘rinishida berilgan bo‘lsin (manbali diffuziya):

$$u(x,0) = \delta(x).$$

Yechimi. Berilgan masalada $\alpha = 1/2$ va $\lambda = 1$. Teorema 1 ga asosan, masalaning yechimi bevosita fundamental yechimning o‘ziga teng bo‘ladi, chunki ixtiyoriy funksiyaning delta-funksiya bilan konvolutsiyasi funksiyaning o‘zini beradi:

$$u(x,t) = \int_{-\infty}^{\infty} G\,(x-\xi,t)\delta(\xi)d\xi = G(x,t).$$

U holda yechim quyidagi ko‘rinishni oladi:

$$u(x,t) = \frac{1}{2t^{1/4}} M_{1/4}\left(\frac{|x|}{t^{1/4}}\right).$$

Bu yerda $M_{1/4}(z)$ funksiyani qator ko‘rinishida quyidagicha yozish mumkin:

$$u(x,t) = \frac{1}{2t^{1/4}} \sum_{k=0}^{\infty} \frac{(-1)^k |x|^k}{k!\, t^{k/4} \Gamma\left(1 - \frac{k+1}{4}\right)}.$$

Ushbu yechim vaqt o‘tishi bilan klassik diffuziyaga qaraganda sekinroq tarqaladi, bu subdiffuziya jarayonining asosiy xususiyatidir.

Yana bir muhim usullardan biri bu — operatorli usul bo‘lib, u yuqoridagi integral almashtirishlarga muqobil hisoblanadi. Ushbu usulning mohiyatini quyidagi misol orqali ko‘rsatamiz.

Misol 2. Quyidagi kasr tartibli tenglama uchun Koshi masalasi berilgan bo‘lsin:

$${}^{C}\partial_{0t}^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = x^2.$$

Yechimi. Ma’lumki, yechimni operator ko‘rinishida $u(x,t) = E_\alpha(t^\alpha \frac{\partial^2}{\partial x^2})x^2$ shaklida tasavvur qilish mumkin. Mittag-Leffler funksiyasining operatorli ko‘rinishini qatorga yoyamiz:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \frac{\partial^{2k}}{\partial x^{2k}} (x^2).$$

Dara jali funksiyadan ketma-ket hosila olamiz:

$$\frac{\partial^0}{\partial x^0}(x^2) = x^2, \quad \frac{\partial^2}{\partial x^2}(x^2) = 2, \quad \frac{\partial^{2k}}{\partial x^{2k}}(x^2) = 0 \quad (\forall k \geq 2).$$

Shunday qilib, cheksiz qator atigi ikkita haddan iborat bo‘ladi: $u(x,t) = \frac{t^0}{\Gamma(1)}x^2 + \frac{t^\alpha}{\Gamma(\alpha+1)} \cdot 2 = x^2 + \frac{2t^\alpha}{\Gamma(\alpha+1)}.$

Bu esa masalaning aniq analitik yechimidir.

Xulosa qilib aytganda, kasr tartibli diffuziya tenglamalari uchun Koshi masalasini yechishda Furye va Laplas integral almashtirishlarining kombinatsiyasi eng samarali va universal apparat hisoblanadi. Olingan analitik yechimlar tahlili shuni ko‘rsatadiki, kasr tartibli modellar tizimning o‘tmishdagi holatlarini (xotirani) saqlab qolish qobiliyatiga ega va tabiatdagi murakkab nomukammal jarayonlarni aniqroq tavsiflaydi.

FOYDALANILGAN ADABIYOTLAR:

1. Podlubny, I. Fractional Differential Equations. Academic Press, San Diego, 1999.
2. Samko, S. G., Kilbas, A. A., Marichev, O. I. Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon, 1993.
3. Kilbas, A. A., Srivastava, H. M., Trujillo, J. J. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam, 2006.
4. Mainardi, F. Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London, 2010.
5. Pskhu, A. V. Uravneniya drobnogo poryadka i ih prilozheniya (Equations of Fractional Order and Their Applications). Nauka, Moscow, 2005. (in Russian).
6. Gerasimov, A. N. "Generalization of linear deformation laws and its application to internal friction problems". PMM, Vol. 12, No. 3, 1948, pp. 251–260.
7. Caputo, M. "Linear models of dissipation whose Q is almost frequency independent—II". Geophysical Journal International, Vol. 13, No. 5, 1967, pp. 529–539.
8. Metzler, R., Klafter, J. "The random walk's guide to anomalous diffusion: a fractional dynamics approach". Physics Reports, Vol. 339, No. 1, 2000, pp. 1–77.
9. Uchaikin, V. V. Method of Fractional Derivatives. Art-Press, Ulyanovsk, 2008.
10. Luchko, Y. "Maximum principle for the generalized time-fractional diffusion equation". Journal of Mathematical Analysis and Applications, Vol. 351, No. 1, 2009, pp. 218–223.